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Calculation of anomalous dimensions in conformally invariant field theory

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Abstract. Within the method developed by us for calculation of the skeleton diagrams in conformally invariant field theory we find anomalous dimensions of fields and the coupling constants in some models of quantum field theory. We also demonstrate the efficiency of the method for finding critical indices in the theory of second-order phase transitions directly in three-dimensional space.

1. Introduction

In this paper we review our results on the determination of anomalous dimensions and of coupling constants in conformal field theory and quantum statistics. Here we dwell only upon the results of the present authors concerning the realisation of the bootstrap programme†. The basis for the study of conformal field theories is provided by the set of completely renormalised equations for the Green functions first obtained by Fradkin (1954, 1955a). It appeared that this set of equations admits conformally invariant solutions (Polyakov 1969, Migdal 1969) which are justified within the conformal theory since the bare terms contain the renormalisation factors which disappear in the present case. This fact was taken as a basis for the bootstrap programme formulated by a number of authors (Migdal 1971, Parisi and Peliti 1971, Mack and Symanzik 1972, Mack and Todorov 1973) which aimed to find coupling constants and anomalous field dimensions. The point is that the set of renormalised equations when represented as an expansion with respect to skeleton diagrams (Migdal 1969) may be reduced to a set of equations solely for two-point Green functions and (in the case of trilinear coupling) three-point vertices. On the other hand, these functions are known (Polyakov 1970) to be determined, up to a normalising factor, by the requirement that the theory be conformally invariant. It is this circumstance which allows one to reduce (if one succeeds in calculating the integrals of skeleton diagrams) the set of integral equations for the two- and three-point Green functions to the set of algebraic equations for finding coupling constants and anomalous dimensions (Migdal 1971, Parisi and Peliti 1971). However, the complications involved in performing the calculations of the integrals which appear in the process of reduction has hindered progress in this direction.

† More details on the present status of conformal field theory may be found in the review article by Fradkin and Palchik (1978).

In the papers by Fradkin *et al* (1977), Palchik (1977) and Zaikin (1978, 1979) a method was proposed to simplify the calculations and this gave us the possibility to advance the realisation of the bootstrap programme and find the coupling constants and anomalous dimensions for various models of conformal field theories and the critical indices in the theory of second-order phase transitions. The appearance of scale invariance and anomalous dimensions has been proposed for the first time by Pataschinskii and Pokrovskii (1964).

The paper is organised as follows. In § 2 we present the description of the method using the simplest case of trilinear interaction as an example. In § 3 the critical indices are found in a model of second-order phase transitions. We consider a space of dimensions $D=4-\epsilon$. The values of the critical indices found coincide with those of Wilson. We also managed to obtain the critical indices directly in three-dimensional space.

The efficiency of the method is demonstrated in § 4, where we solve a number of complex models of the triple interaction for both Bose and Fermi particles.

2. Method for calculating anomalous dimensions

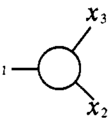
Let us take as an example the simplest scalar field model with

$$\mathscr{L}_{\text{int}}=\lambda\phi^3/3! \tag{1}$$

in D-dimensional space†. The solution will be given for $D=6$.

Define the normalisation of the conformally invariant propagator and vertex as

$$G(x_1x_2x_3)=\langle 0|T(\phi_d(x_1)\phi_d(x_2)\phi_d(x_3))|0\rangle =$$

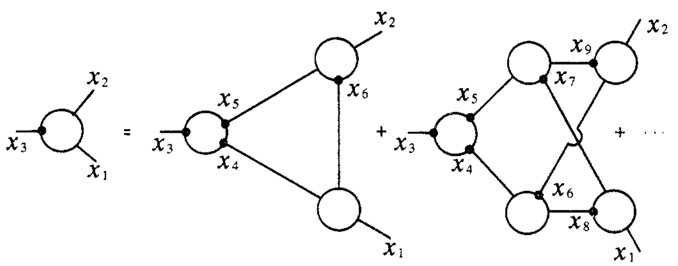


$$=g(x_{12}^2x_{23}^2x_{31}^2)^{-d/2} \qquad x_{ij}^2=(x_i-x_j)^2$$

$$G(x_1x_2)=\langle 0|T(\phi_d(x_1)\phi_d(x_2))|0\rangle=(1/\pi^h)(\Gamma(d)/\Gamma(h-d))(x_{12}^2)^{-d}$$

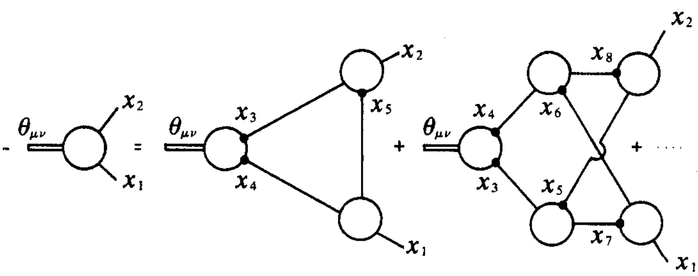
where g is a coupling consant and d is the anomalous dimension of the scalar field $\phi_d(x)$; $h=\frac{1}{2}D$.

To calculate the values of g and d we shall take the renormalised equations, written as skeleton equations (bootstrap equations):



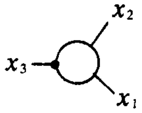
(2)

† The Eudlidian formulation of quantum field theory (Fradkin 1959, Schwinger 1959, Nakano 1959) is used throughout the paper.



(3)

A dot on the line entering a vertex shows the amputation of vertices with respect to the corresponding leg. For instance,

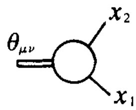


$$= \int G(x_1x_2x_3')G^{-1}(x_3'x_3)\,\mathrm{d}^Dx_3'$$

$$= g(\Gamma(\tfrac{1}{2}(D-d)/\Gamma(\tfrac{1}{2}d))^2(x_{12}^2)^{(D-3d)/2}(x_{13}^2x_{23}^2)^{-(D-d)/2}.$$

Calculations of the amputated vertices are performed within the method proposed by Symanzik (1972) and Ferrara *et al* (1972, 1974) and developed by Fradkin and Palchik (1974, 1975a, b). The equation for the propagator (3) is taken in the form proposed by Mack and Symanzik (1972).

The designation



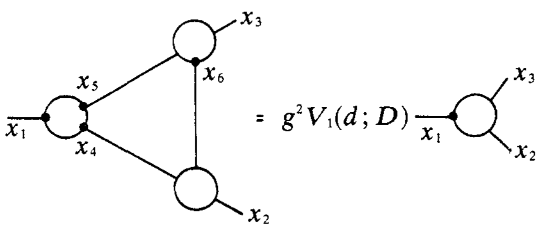
stands for the three-point vertex of the energy–momentum tensor $\theta_{\mu\nu}$ and two scalar fields

$$G_{\mu\nu}(x_1x_2x_3) = \langle 0|T(\theta_{\mu\nu}(x_3)\phi(x_1)\phi(x_2))|0\rangle.$$

This vertex satisfies the generalised Ward identity (Fradkin 1955b, Takahashi 1957)

$$-\partial_\mu^{x_3}G_{\mu\nu}(x_1x_2x_3) = [\delta(x_3-x_1)\partial_\nu^{x_1} + \delta(x_3-x_2)\partial_\nu^{x_2}$$
$$-(d/D)\partial_\nu^{x_3}\delta(x_3-x_1) - (d/D)\partial_\nu^{x_3}\delta(x_3-x_2)]G(x_1x_2). \qquad (4)$$

The right-hand sides of equations (2) and (3) are infinite sums of graphs which are two-particle irreducible in the longitudinal direction. The statement about the conformal invariance of the equations made in the introduction refers not only to the whole sum but also to each term on the right-hand sides of these equations. Consequently, each term of these sums transforms in the same way as the three-point function; because the latter is defined uniquely, each term is proportional to the left-hand side of the corresponding equations. The proportionality coefficient is a function of the dimensions of the fields and the space. For instance,



(5)

Now we come to the set of algebraic equations we spoke about:

$$1 = g^2 V_1(d; D) + g^4 V_2(d; D) + \ldots \tag{6a}$$

$$1 = g^2 P_1(d; D) + g^4 P_2(d; D) + \ldots. \tag{6b}$$

Let us solve these equations by perturbation, assuming that we may confine ourselves in equations (6) to the first term of the series in powers of g^2. Hence it follows that at least the following inequalities must hold:

$$g^2 V_2(d; D)/V_1(d; D) \ll 1 \qquad g^2 P_2(d; D)/P_1(d; D) \ll 1. \tag{7}$$

The simplest way to satisfy both inequalities is to look for solutions of equations (6) for which $g^2 \ll 1$. If we simultaneously confine ourselves, say, to the three-vertex approximation we necessarily have $P_1(d; D) \gg 1$, $V_1(d; D) \gg 1$ (since in this case $g^2 V_1 = g^2 P_1 = 1$). This requirement only strengthens the inequalities (7). Consequently, we should look for the solution with $g^2 \ll 1$ in the vicinity of the common poles in d of the functions $V_1(d; D)$ and $P_1(d; D)$. The location of these poles may be found as follows.

Let us first consider the three-vertex term. It is convenient to integrate both sides of (5) over x_1:

$$g^2 V_1(d; D)(x_{23}^2)^{-d/2}$$
$$= g^3 \frac{\Gamma^3(\frac{1}{2}(D-d))\Gamma(D-\frac{3}{2}d)}{\Gamma^3(\frac{1}{2}D)\Gamma(\frac{1}{2}(3d-D))} \int (x_{24}^2 x_{26}^2 x_{46}^2)^{-d/2} (x_{36}^2 x_{56}^2)^{-(D-d)/2} (x_{35}^2)^{-(3d-D)/2}$$
$$\times (x_{45}^2)^{-(2D-3d)/2}\, \mathrm{d}^D x_4\, \mathrm{d}^D x_5\, \mathrm{d}^D x_6. \tag{8}$$

In order to find the positions of the poles of this expression, we recall that the function $(x^2)^{-\alpha}$ treated as a function of the parameter α has first-order poles at $\alpha = \frac{1}{2}D + k$, $k = 0, 1, 2, \ldots$. It is known (Gelfand and Shilov 1964) that

$$\lim_{\epsilon\to 0} \epsilon (x^2)^{-(h+k-\epsilon)} = \frac{\Omega_D}{2} \frac{\Gamma(h)}{\Gamma(h+k)} \frac{4^{-k}}{\Gamma(k+1)} \Box^K \delta(x) \tag{9}$$

where

$$\Omega_D = 2\pi^h/\Gamma(h).$$

Also, the expression $(x_{12}^2)^{-\alpha}(x_{23}^2)^{-\beta}(x_{31}^2)^{-\gamma}$ has first-order poles in the variables $S = \alpha + \beta + \gamma$ at $S = D + m$, $m = 0, 1, 2 \ldots$ ($\alpha, \beta, \gamma \neq 0; \frac{1}{2}D + l; l = 0, 1, 2 \ldots$). By using the Fourier transform of the given expression one can prove that

$$\lim_{\epsilon\to 0} \epsilon (x_{12}^2)^{-\alpha}(x_{23}^2)^{-\beta}(x_{31}^2)^{-\gamma}|_{\gamma = D-\alpha-\beta-\epsilon}$$
$$= \left(\frac{\Omega_D}{2}\right)^2 \Gamma(h) \frac{\Gamma(h-\alpha)\Gamma(h-\beta)\Gamma(\alpha+\beta-h)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)} \delta(x_{12})\, \delta(x_{23}). \tag{10}$$

Using equations (9) and (10) one can find the locations of all the poles of the integrand in (8) and their residues. In the following we shall confine our search to solutions of equations (6) near the simplest pole of the function $V_1(d; D)$, namely $d = \frac{2}{3}D$. Put

$$d = \tfrac{2}{3}D + \tfrac{2}{3}\epsilon_1.$$

Now we readily find the first and the second orders in ϵ_1 in the expansion of the function $V_1(d; D)$ and the first order in ϵ_1 for the function $V_2(d; D)$. Expansions of the functions $P_1(d;D)$ and $P_2(d;D)$ induced by the equation for the energy–momentum-tensor-containing vertices (3) are found analogously. The expansion of $P_1(d;D)$ may also be found by using the exact (valid for all d and D) expression for P_1 obtained by Fradkin and Palchik (1975b). It is worth mentioning that the expansion of the function $P_1(d, D)$ depends essentially on the dimensionality D of the space. Put $D=6$. Then the algebraic set acquires the form (for more details see Palchik 1977)†

$$1=-\frac{1}{\epsilon_1^3}g_1^2[1-\epsilon_1(\tfrac{7}{2}+4\psi(1))]-g_1^4\frac{2}{\epsilon_1^5} \tag{11a}$$

$$1=-\frac{1}{\epsilon_1^3}g_1^2[1-\epsilon_1(4\psi(1)+\tfrac{55}{18})]-g_1^4\frac{1}{12\epsilon_1^5} \tag{11b}$$

where

$$g_1^2=g^2(\tfrac{1}{2}\Omega_D)^2 \qquad \psi(x)=\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x).$$

These equations are accurate when $\epsilon_1 \ll 1$. It can easily be seen that in the three-vertex approximation (i.e. when confining ourselves only to the leading terms on the right-hand side) there are no solutions. However, in the five-vertex approximation there is a solution with small ϵ_1:

$$g_1^2\approx -1{\cdot}7\times 10^{-3} \qquad \epsilon_1\approx 0{\cdot}07 \qquad d=(4+\tfrac{14}{3})\times 10^{-2}. \tag{12}$$

It follows already from the three-vertex approximation that $g^2\sim\epsilon_1^3$, and thus the true expansion parameter is ϵ_1, i.e. the deviation of the dimension d from $d=4$. For this reason we kept the next order in ϵ_1 in the three-vertex term and only the leading order in the five-vertex term.

3. Critical indices in a model of second-order phase transitions

Consider now the interaction Lagrangian

$$\mathscr{L}_{\text{int}}=\lambda\phi^4. \tag{13}$$

This gives the model for phase transitions of second order. Wilson and Fisher succeeded in using the ϵ expansion in the $(4-\epsilon)$-dimensional space to find the anomalous dimensions of the fields and critical indices in this model. Below an alternative method is proposed for obtaining anomalous dimensions in this model which work not only in the space of $D=4-\epsilon$ dimensions but directly in three-dimensional space.

Suppose that the theory given by (13) is conformally invariant. To exploit this assumption efficiently, it is useful to formulate the interaction (13) in terms of the auxiliary field

$$\chi(x)\sim :\phi^2(x):$$

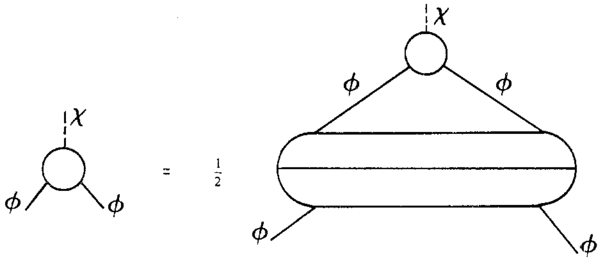
The new Lagrangian is of the form:

$$L_{\text{int}}=\tfrac{1}{2}g\phi^2\chi.$$

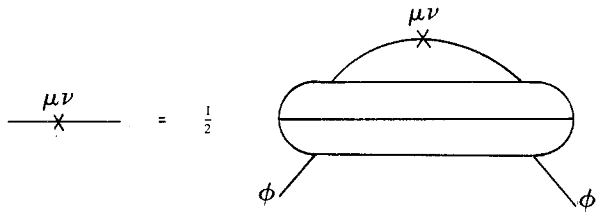
† In this paper another normalisation of the vertex and propagator has been chosen, so (11*a*, *b*) are slightly different from the analogous equations in the paper by Palchik (1977).

Let d be the dimension of the fundamental field $\phi(x) \equiv \phi_d(x)$, and Δ be the dimension of the auxiliary field $\chi(x) \equiv \chi_\Delta(x)$. We shall take the renormalised skeleton equations for vertices and propagators as a starting point.

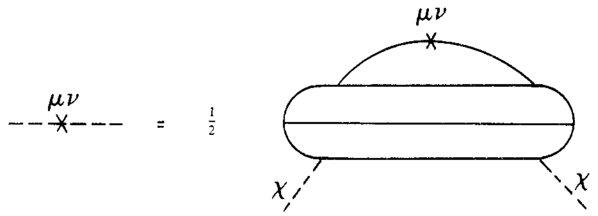
In the present case it is convenient to take them in the form:



(14*a*)

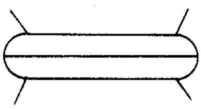


(14*b*)

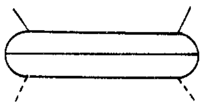


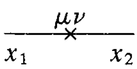
(14*c*)

where



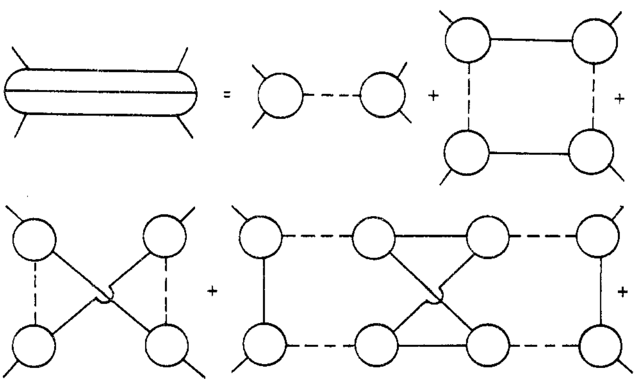
and





$\equiv (x_1 - x_2)_\mu \, \partial_\nu^{x_1} G^{-1}; \qquad \mu \neq \nu$

are Bethe–Salpeter kernels which cannot be divided into two parts in the longitudinal channel by cutting two ϕ lines. These kernels may be represented as a skeleton decomposition. For instance,



+ cross terms + ... (15)

Assume that the dimensions of the conformal vertices and propagators lie near the values $\frac{1}{4}D$ and $\frac{1}{2}D$:

$$d=\tfrac{1}{4}D-\epsilon_1 \qquad \Delta=\tfrac{1}{2}D-\epsilon_2 \tag{16}$$

where ϵ_1 and ϵ_2 are small, and let us find the solution. These values are special (see below) since some skeleton graphs on the right-hand sides of equations (14) have poles at $\epsilon_1=\epsilon_2=0$ as functions of the dimensions d and Δ. This leads to the situation indicated in the previous section: on one hand it suffices to keep only these graphs since they give the leading contribution; on the other hand, one may use the methods developed in the previous section in order to compute them. This is essentially valid in the space of any dimension. Note that at $D=4-\epsilon$ the dimensions (16) lie near their canonical values $d=\frac{1}{4}D-\epsilon_1=(\frac{1}{2}D-1)+\frac{1}{4}\epsilon-\epsilon_1$. In the three-dimensional space the value of d is $d=\frac{3}{4}-\epsilon_1$ and is far from its canonical value. The results of Pataschinskii and Pokrovskii (1964) correspond to the zero approximation in ϵ_1 and ϵ_2.

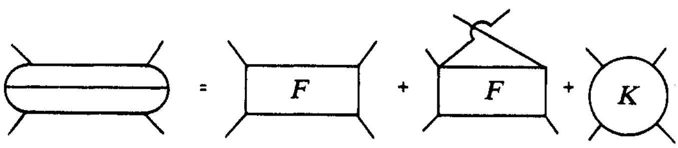
We present here as an example the result calculating the first two graphs on the right-hand side of (14*a*) for the dimensions (16). Keeping only leading terms with respect to ϵ and using equations (9) and (10) we obtain

$$\text{three-vertex contribution}=-g^2\frac{2\epsilon_1}{\frac{1}{2}(\epsilon_2-2\epsilon_1)\frac{1}{2}(\epsilon_2+2\epsilon_1)}$$

$$\text{five-vertex contribution}=-g^4\frac{2}{\frac{1}{2}(\epsilon_2-2\epsilon_1)\frac{1}{2}(\epsilon_2+2\epsilon_1)}.$$

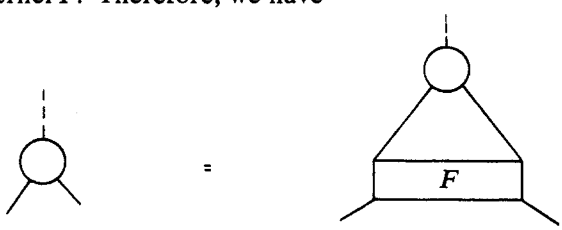
Both these terms must be kept since ϵ_1, ϵ_2 and g^2 are parameters of the same order of magnitude (this follows from (14*c*); see Fradkin and Palchik (1978)). Moreover, there is also an infinite set of diagrams (analogous to the parquet graphs) which are of the order of $(g^2/\epsilon')^h$, where $\epsilon'\sim\epsilon_1\sim\epsilon_2$, and consequently give the same contribution. All these diagrams must be summed.

The problem of summing them may be solved by a method analogous to the one developed by Sudakov (1956), Dyatlov *et al* (1957), Pomeranchuk *et al* (1956) and Patashinskii and Pokrovskii (1975) for summing the parquet diagrams. Let us represent the Bethe–Salpeter kernel as



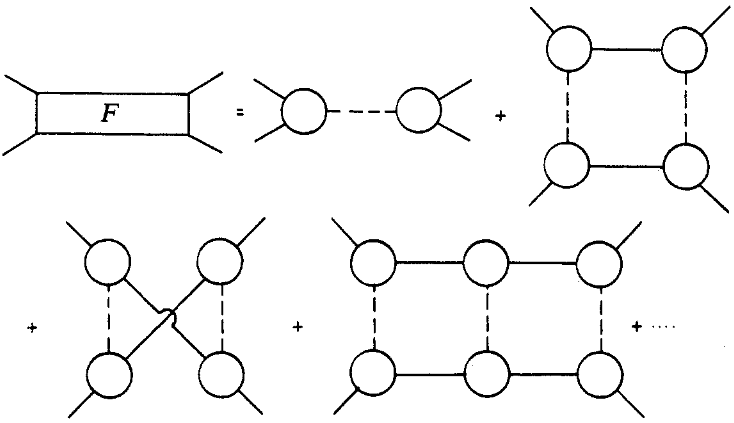
(17)

Here F includes the first graph from (15) and the sum of all graphs which can be separated by making two vertical full-line cuts and K is the sum of graphs which cannot be separated by making two full-line cuts in any direction (K includes, in particular, the fourth graph from (15)). Let us substitute (17) into (14*a*) and keep only those graphs which create poles in the point (16). One can easily see that such graphs are contained only in the kernel F. Therefore, we have



(17*a*)

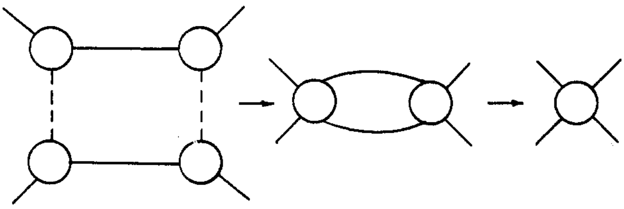
Further, of all graphs contained in F one must keep only the reducible ones (in the terminology of the paper by Pomeranchuk *et al* (1956)):



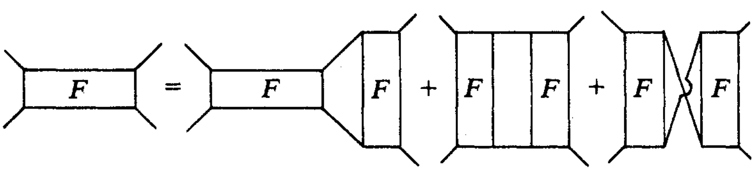
(17b)

Here we keep only those graphs (the reducible ones) which can be converted into the vertex

by making (i) every pair of vertices joined by a dotted line and (ii) every pair of vertices joined by two full lines stick together. For instance:



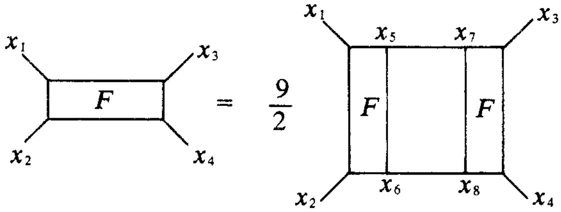
The remaining graphs do not give poles in the points (16). It is essential to note that the quantity F defined in (17b) is a sum of skeleton graphs and thus differs from the analogous quantity in the papers mentioned above, wherein Feynman graphs are summed. One may show that F obeys the equation



(17c)

To this end it suffices to substitute (17b) into (17c) and to utilise the bootstrap equation (17a) (to deduce the first term on the right-hand side).

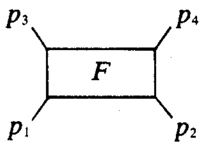
To suit our purposes, however, it is more convenient to use, instead of (17c), the equation for the kernel F, taken at coinciding external momenta (its derivation is explained below):



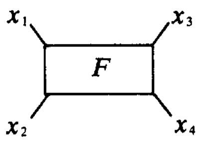
(18)

In deriving equation (18) it is assumed that all the coordinate differences are of the same order of magnitude and that the leading contribution in the integral comes from the region $x_{56} \to 0$ (or $p \to 0$ in momentum space). This equation is obtained by Pomeranchuk *et al* (1956) for $D = 4$. One readily sees that in our case this remains valid for any space dimension provided ϵ_1 and ϵ_2 are sufficiently small. Indeed, the authors of the papers mentioned above essentially used the presence of logarithmic divergences, and the derivation of equation (18) was reduced to summing the leading logarithms (which are contained in the parquet graphs). In our case, because of the special choice of dimension (16), exactly the same situation arises in the ultraviolet region for any space dimension. At $\epsilon_1 = \epsilon_2 = 0$ (i.e. at the poles) the usual ultraviolet divergencies are reproduced. The leading singularities arise in the skeleton graphs of the parquet type while the role of the large logarithm (to be more precise, of $\lambda \ln(p^2/\Lambda)$) is played by the quantity g^2/ϵ', where $\epsilon' \sim \epsilon_1 \sim \epsilon_2$ (recall once again that this concerns the *skeleton* graphs (17*b*), which only in the particular case $D = 4$ reduce to the usual Feynman ones). Thus, the whole argument on the summation of Feynman graphs is readily extended, without any changes, to the summing of skeleton graphs with scale-invariant dimension (16) and with any space dimension.

For what follows the most important point is the fact that the quantity

$F(p_1 p_2 p_3 p_4) =$ 

depends only on a single large momentum, if the conditions $p_1, p_2 \sim p$, $p \gg p_3, p_4$ are fulfilled. In this case $F = F(p) \sim (p^2)^{h-2d}$. In the configuration space the condition $p \gg p_3 p_4$ means that $x_{13} \to 0$. The corresponding expression for F is

$F(x_1 x_2 x_3 x_4) =$ 

$$= f\delta(x_{12})\,\delta(x_{34})(x_{13}^2)^{-(D-2d)} \qquad (19)$$

where $x_{13} \to 0$ or $x_{24} \to 0$. If, on the contrary, all the coordinate differences are of the same order (correspondingly $p_1 \sim p_2 \sim p_3 \sim p_4$), i.e. $\epsilon_{12} \ln x_{12}^2 \ll 1$, we must replace $(x_{13}^2)^{-D+2d}$ by the limiting expression $\pi^h f/(2\epsilon_1 \Gamma(h))\,\delta(x_{13})$ (see (9)). Then in place of (19) we have

$$F(x_1 x_2 x_3 x_4) = \frac{\pi^h}{\Gamma(h)} f \frac{1}{2\epsilon_1}\,\delta(x_{12})\,\delta(x_{14})\,\delta(x_{13}). \qquad (19a)$$

Let us now substitute (19*a*) into the left- and (19) on the right-hand side of (18). After the calculation of the integrals on the right-hand side we must go to the limiting expression like (19*a*) by using equation (9). This results in the following expression for the 'coupling constant':

$$f=f^2\frac{\Gamma^2(d)}{\Gamma^2(h-d)}\cdot\frac{1}{\Gamma^2(h)}\frac{9}{8}\frac{1}{(\frac{1}{4}D-d)^2}. \tag{20}$$

After the substitution of (19) and (20) into (17*a*) and (14*b*) we obtain the algebraic equations for the dimensions:

$$\begin{aligned} &1=\bar{f}\frac{\Gamma^2(h-d)\Gamma(d+\frac{1}{2}\Delta-h)\Gamma(d-\frac{1}{2}\Delta)}{\Gamma^2(d)\Gamma(D-d-\frac{1}{2}\Delta)\Gamma(h-d+\frac{1}{2}\Delta)}\\ &1=-\bar{f}\frac{d}{D-d}\frac{\Gamma(h-d)\Gamma(d-h)}{\Gamma(d)\Gamma(D-d)}\\ &1=\bar{f}\frac{1}{\Gamma^2(h)}\frac{9}{8}\frac{1}{(\frac{1}{4}D-d)^2}\qquad \bar{f}=f\frac{\Gamma^2(d)}{\Gamma^2(h-d)}. \end{aligned} \tag{21}$$

These equations are valid in the main order in ϵ_1 and ϵ_2 for any space dimension if ϵ_1 and ϵ_2 are sufficiently small. When $D=4-\epsilon$ the substitution of $d=1-\frac{1}{2}\epsilon+C_0\epsilon^2$ and $\Delta=h-C_1\epsilon$ into (21) leads to the known solution (Wilson and Kogut 1947):

$$d=\tfrac{1}{2}D-1+\tfrac{1}{108}\epsilon^2\qquad \Delta=\tfrac{1}{2}D-\tfrac{1}{6}\epsilon.$$

If $D=3$ the approximate solution of the set of equations (21) has the form

$$d\approx\tfrac{1}{2}+0{\cdot}01\qquad \Delta\approx\tfrac{3}{2}-0{\cdot}17. \tag{22}$$

A knowledge of the dimensions of the fields ϕ and χ makes it easy to find the critical indices η and ν. The index η characterises the deviation of the behaviour of the two-point Green function $G(x_1x_2)$ from that of the free one. The index ν is associated with the singular behaviour of the correlation radius distance near the phase-transition point:

$$\xi\sim\left|\frac{T-T_c}{T_c}\right|^{-\nu}\qquad \nu=\frac{1}{D-\Delta}\approx 0{\cdot}60.$$

The remaining critical indices may be found from the scaling relations that connect these indices with the known η and ν. The method described here can be applied to a rather wide class of models in any space–time dimensions. It is important that in this case scale dimensions of fields can turn out to be far from their canonical values. From this point of view the model of four-fermion interaction is of special interest and it will be considered elsewhere.

Note that for a sufficiently wide class of theories and with any dimensionality of the space one may always find such field dimensions (perhaps far from their canonical values) that all the parquet skeleton graphs turn out to be of the same order. It would be of interest to apply the method developed to four-fermion interaction in four-dimensional space. We hope to do this elsewhere.

4. Anomalous dimensions in models of triple interactions

Consider more models and find anomalous field dimensions by applying the method described in § 2.

4.1. *Model 1*

$$\mathcal{L}_{\text{int}} = \frac{g}{2}\left(\sum_{i=1}^{N} \phi_i^2\right)\chi. \tag{23}$$

Here ϕ_i is an N-component scalar field with the dimension d_ϕ and χ is a one-component scalar field with the dimension d_χ.

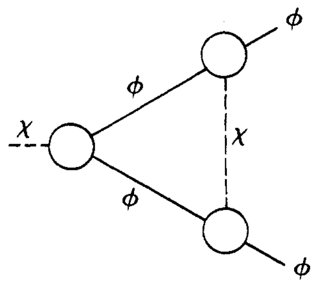
We will search for a conformally invariant solution of this model in the space of $D = 6+\epsilon$ dimensions near the canonical values of the field dimensions. Assume that the deviation from the canonical values is of order ϵ:

$$d_\phi = \tfrac{1}{2}D - 1 + C_\phi\varepsilon + \ldots = 2 - \tfrac{1}{2}\epsilon + C_\phi\epsilon + \ldots \tag{24}$$

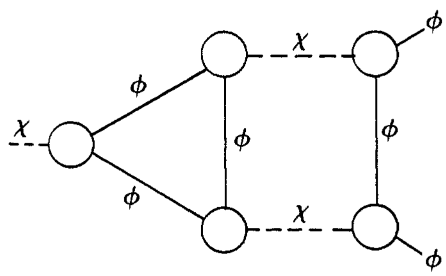
$$d_\chi = \tfrac{1}{2}D - 1 + C_\chi\epsilon + \ldots = 2 - \tfrac{1}{2}\epsilon + C_\chi\epsilon + \ldots. \tag{25}$$

The bootstrap equation for the vertex $G(x_1x_2x_3) = \langle\phi(x_1)\phi(x_2)\chi(x_3)\rangle$ coincides in form with the one considered in the previous section (see equation (14*a*)). Now, however, we must separate in the Bethe–Salpeter kernel another set of graphs which gives a contribution of the same order when the expansion in powers of ϵ is considered.

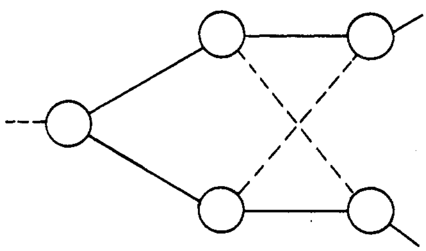
One may readily make sure that the diagrams



and



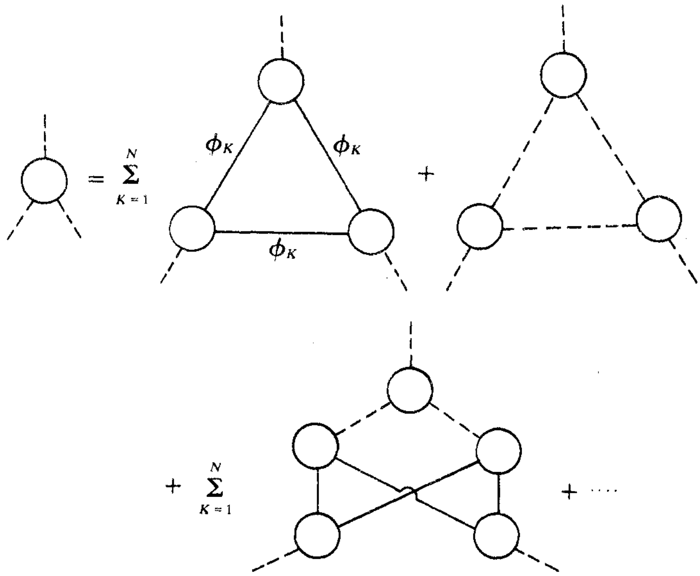
are, near the dimensions (24), to first order in ϵ proportional to $g(g^2/\epsilon)$ and $g(g^4/\epsilon^2)$. As for the equation for the propagator of the χ field, it shows that $g^2 \sim \epsilon$ and therefore the above diagrams are of equal order of magnitude. Note that the diagram



is of order $g(g^4/\epsilon)$ and, consequently, is of first order in ϵ.

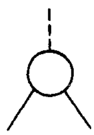
One may easily see that to higher orders of g^2 diagrams appear which are of the same order in ϵ as the three-vertex contribution. All these diagrams may be summed in the same way as was done in § 3. In the present case the summation is considerably easier.

It is sufficient to introduce into the consideration the vertex of three χ fields whose equation is

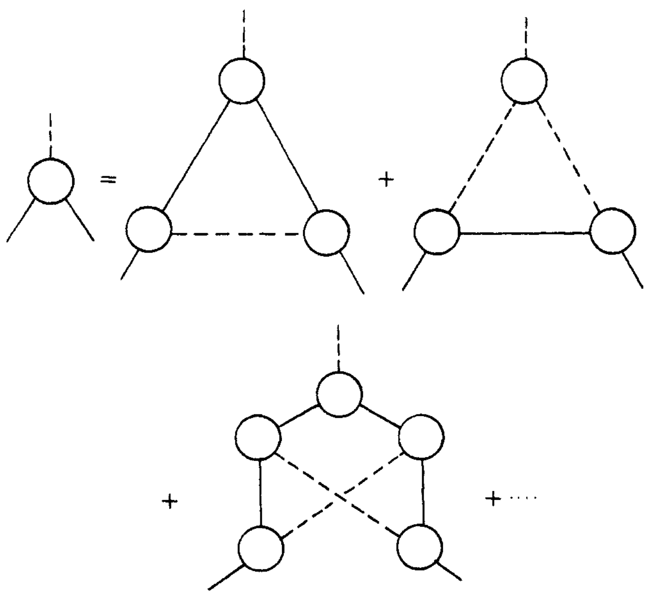


(26)

Then the equation for the vertex



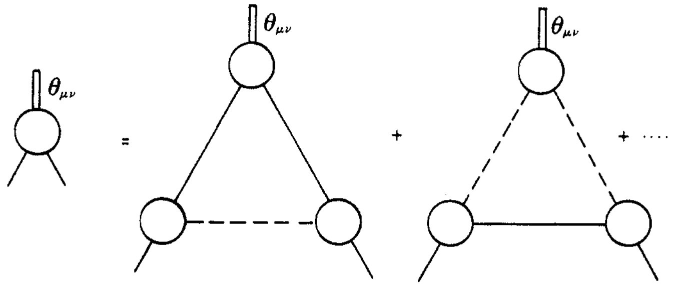
transforms to the form



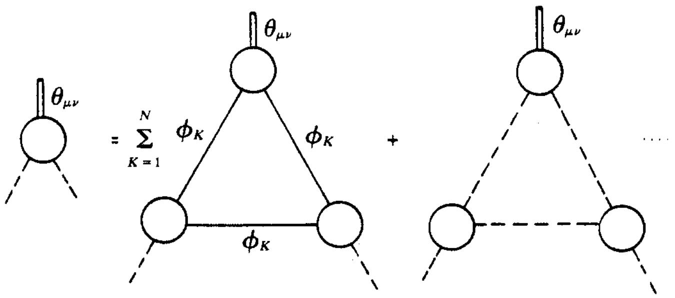
(27)

Now in all the equations the five-vertex graphs are already of the next order in ϵ as compared with the three-vertex ones.

Equations for the propagators of the fields ϕ and χ should be also written using the newly introduced vertex of the three fields χ:



(28)



(29)

The set of equations (26)–(29) obtained determines the four unknown C_ϕ, C_χ, g and λ, the latter being the coupling constant of the vertex of the three scalar fields χ.

The method of § 2 makes it easy to write the corresponding algebraic equations. Within the three-vertex approximation they are·

$$1=\frac{1}{\epsilon}(g_1^2+g_1\lambda_1)\frac{2}{(\frac{1}{2}+2C_\phi+C_\chi)}+\ldots$$

$$1=\frac{1}{\epsilon}\left(Ng_1^2\frac{g_1}{\lambda_1}+\lambda_1^2\right)\frac{1}{(\frac{1}{2}+3C_\chi)}+\ldots$$

$$1=\frac{1}{\epsilon}(Ng_1^2+\lambda_1^2)\frac{1}{6C_\chi}+\ldots \tag{30}$$

$$1=\frac{1}{\epsilon}g_1^2\frac{1}{3C_\phi}+\ldots$$

where

$$g_1^2=(\tfrac{1}{2}\Omega_D)^3\Gamma^3(3)g^2 \qquad \lambda_1^2=(\tfrac{1}{2}\Omega_D)^3\Gamma^3(3)\lambda^2.$$

The investigation of this set of equations shows that for $\epsilon>0$ there are solutions obeying the conditions $\lambda^2>0$, $g^2>0$, $d_\phi>\frac{1}{2}D-1$, $d_\chi>\frac{1}{2}D-1$ only when $N=1$. If $N=1$ the following two solutions exist:

$$g_1^2=\lambda_1^2=\tfrac{1}{6}\epsilon \qquad C_\phi=C_\chi=\tfrac{1}{18} \tag{31a}$$

$$g_1^2=\tfrac{75}{499}\epsilon \qquad \lambda_1^2=\tfrac{108}{499}\epsilon \qquad C_\phi=\tfrac{25}{499} \qquad C_\chi=\tfrac{661}{499}. \tag{31b}$$

The solution (31a) corresponds to the degenerate case when the fields ϕ and χ are indistinguishable and the interaction is, as a matter of fact, reduced to $g\phi^3/3!$. In this degenerate case the result coincides with that of Mack (1972).

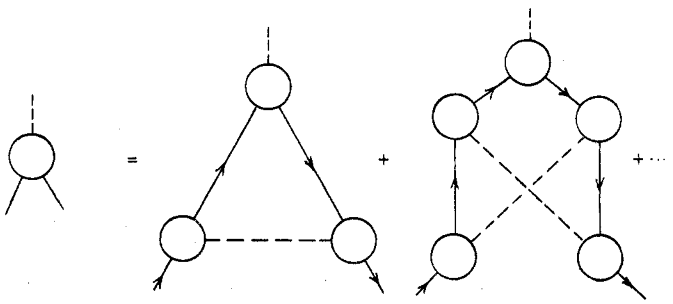
4.2. *Model 2*

$$\mathscr{L}_{\text{int}} = g\left(\sum_{i=1}^{N} \bar{\psi}_i\psi_i\right)\phi. \tag{32}$$

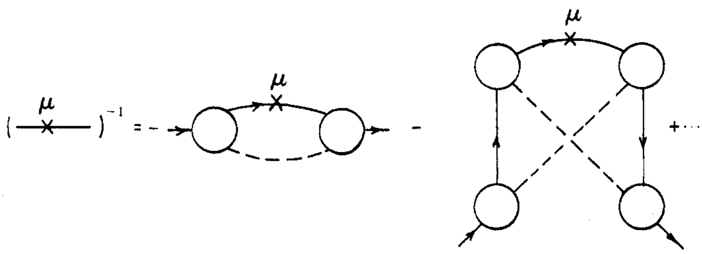
The technique of calculation developed in § 2 may be applied almost unaltered to the Yukawa model. We assume that ψ is an N-component spinor field whose dimension is d and ϕ is a scalar field with the dimension Δ. We study this model in the space of dimensionality $D = 4-\epsilon$. We search for solutions for the field dimensions near their normal (canonical) values:

$$d = \tfrac{1}{2}(D-1) + C_1\epsilon + \ldots \qquad \Delta = \tfrac{1}{2}D - 1 + C_2\epsilon + \ldots. \tag{33}$$

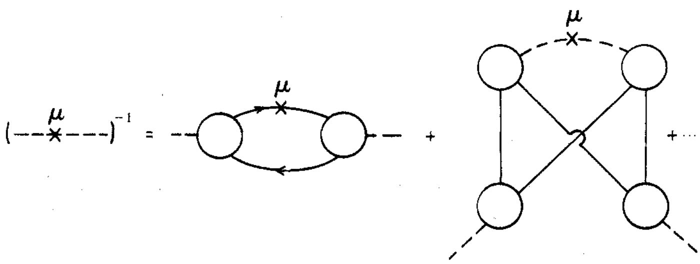
We confine ourselves to leading order in ϵ. The set of bootstrap equations for this interaction is readily written as



(34a)

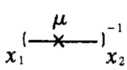
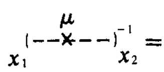


(34b)



(34c)

where

 $= (x_{12})_\mu G^{-1}(x_1x_2)$  $= (x_{12})_\mu D^{-1}(x_1x_2)$.

The quantity $G(x_1x_2; x_3) = \langle T(\psi_j(x_1)\bar{\psi}_j(x_2)\phi(x_3))\rangle$ and the spinor propagator $G(x_1x_2) = \langle T(\psi_j(x_1)\bar{\psi}_j(x_2))\rangle$ (no summation over repeated indices is assumed) are chosen

$$G(x_1x_2; x_3) = g\hat{x}_{13}(x_{31}^2)^{-(\Delta+1)/2}\hat{x}_{32}(x_{32}^2)^{-(\Delta+1)/2}(x_{12}^2)^{-(2d-\Delta)/2}$$

$$G(x_1x_2) = -\mathrm{i}\pi^{-h}\Gamma(d+\tfrac{1}{2})/\Gamma(h-d+\tfrac{1}{2})\hat{x}_{12}(x_{12}^2)^{-(2d+1)/2}$$

where $\hat{x} = x_\mu\gamma^\mu$. For calculations we shall need a formula analogous to (10):

$$\lim_{\epsilon\to 0} \epsilon \frac{\hat{x}_{13}}{(x_{13}^2)^{\alpha+\frac{1}{2}}} \frac{\hat{x}_{32}}{(x_{32}^2)^{\beta+\frac{1}{2}}} (x_{12}^2)^{-(D-\alpha-\beta-\epsilon)}$$

$$= (\tfrac{1}{2}\Omega_D)^2\Gamma(\tfrac{1}{2}D)\frac{\Gamma(h-\alpha+\frac{1}{2})\Gamma(h-\beta+\frac{1}{2})\Gamma(\alpha+\beta-h)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(D-\alpha-\beta)}\delta(x_{12})\,\delta(x_{23}). \qquad (35)$$

The exact expressions used to calculate the three-vertex contributions on the right-hand sides of equations (34*b*) and (34*a*) are given in the appendix. All the integrations are performed in full analogy with § 2. One readily establishes that the five-vertex terms contribute as ϵ^2 to the values of the anomalous dimensions. They will therefore be neglected. The resulting algebraic set of equations

$$1 = \frac{2g^2}{\epsilon(\frac{1}{2} - 2C_1 - C_2)} \qquad 1 = g^2/(2C_1\epsilon) \qquad 1 = 2Ng^2/(C_2\epsilon) \qquad (36)$$

is solved to give

$$d = \frac{D-1}{2} + \frac{\epsilon}{4(2N+3)} \qquad \Delta = \frac{D}{2} - 1 + \frac{N}{2N+3}\epsilon \qquad g^2 = \frac{\epsilon}{2(2N+3)}. \qquad (37)$$

This result coincides with those of Ferrara and Giafaloni (1975), Geicke and Meyer (1973) and Hu (1974, 1975).

The same model (32) may be considered directly in three-dimensional space.

Let us look for solutions of the bootstrap equations (34) near the dimensions of the fields (Zaikin 1978):

$$d = \tfrac{1}{2}(D-1) + \epsilon_0 \qquad \Delta = D - 2d - 2\epsilon_1 \qquad |\epsilon_1|, \epsilon_0 \ll 1. \qquad (38)$$

Note that the dimensions of the fields intrinsic to the solutions found in $D = 4 - \epsilon$ space are also connected by condition (38):

$$\epsilon_1 = \tfrac{1}{2}(D - 2d - \Delta) = \tfrac{1}{4}(1 - C_1 - \tfrac{1}{2}C_2)\epsilon \qquad \epsilon_1 \to 0 \text{ if } \epsilon \to 0.$$

The algebraic set of equations in the three-dimensional space is

$$1 = g_1^2/\epsilon_1 \qquad 1 = g_1^2/3\epsilon_0 \qquad 1 = \tfrac{3}{4}\pi N g_1^2. \qquad (39)$$

The field dimensions and the expansion parameters are found from these to be

$$d = 1 + \frac{4}{9\pi^2 N} \qquad \Delta = 1 - \frac{32}{9\pi^2 N} \qquad g_1^2 = \frac{4}{3\pi^2}\frac{1}{N}$$

$$\epsilon_1 = \frac{4}{3\pi^2 N} \qquad \epsilon_0 = \frac{4}{9\pi^2 N}. \qquad (40)$$

ϵ_1 and ϵ_0 appear to be sufficiently small even at $N = 1$.

Within the model considered we are able to compare the results of calculations performed in three-dimensional space with those for $D = 4 - \epsilon$ dimensions with ϵ put equal to unity:

$$d_{4-\epsilon}-d_3=\frac{1}{4(2N+3)}-\frac{4}{9\pi^2 N}\qquad \Delta_{4-\epsilon}-\Delta_3=\frac{N}{2N+3}-\left(\frac{1}{2}-\frac{32}{9\pi^2 N}\right).$$

When $N=1$ these differences are given by

$$\mathrm{d}_{4-\epsilon}-\mathrm{d}_3\approx 0{\cdot}005\qquad \Delta_{4-\epsilon}-\Delta_3\approx 0{\cdot}06. \tag{41}$$

The analytical continuation with respect to ϵ in the limit $N\to\infty$ gives correctly the functional dependence on N, but not the coefficient in front of $1/N$ in the expressions for the field dimensions. The asymptotic behaviour of the dimensions d and Δ in the spaces $D=4-\epsilon$ and $D=3$ at $N\to\infty$ are

$$\begin{aligned} d_{4-\epsilon}&=1+\frac{1}{8N}\Big|_{\epsilon=1} & d_3&=1+\frac{4}{9\pi^2}\frac{1}{N}\\ \Delta_{4-\epsilon}&=1-\frac{3}{4}\frac{1}{N}\Big|_{\epsilon=1} & \Delta_3&=1-\frac{32}{9\pi^2}\frac{1}{N}. \end{aligned} \tag{42}$$

4.3. Model 3

The problem of finding anomalous field dimensions for the Yukawa-type model directly in four-dimensional space is of great importance for conformal quantum field theory. Consider the theory with the interaction Lagrangian

$$\mathscr{L}_{\text{int}}=\mathrm{i}g\bar{\psi}_i(x)\gamma^5\lambda^a_{ij}\psi_j(x)\phi_a(x) \tag{43}$$

where the λ^a_{ij} are generators of the SU(N) group.

For this interaction there are no solutions near the normal values of the dimensions of the fields ψ and ϕ (Galanin 1975). Therefore, the dimension of at least one of the fields should be far from its canonical value (Zaikin 1978, 1979).

Set

$$d=\tfrac{1}{2}(D-1)+\epsilon_0\qquad \Delta=2d+\epsilon_1. \tag{44}$$

One sees from (44) that the dimension of the scalar field Δ deviates considerably from its canonical value ($\Delta_{\text{canonical}}=\frac{1}{2}\mathrm{D}-1$). The bootstrap equations for the interaction (43) coincide with equations (34). The form of the propagators and the vertex function is trivially generalised for the case of the SU(N) group.

The set of algebraic equations may be obtained easily by the method developed as

$$1=-g_0^2\frac{\pi^4}{\epsilon_1}\frac{2}{N}\qquad 1=g_0^2\frac{\pi^4}{\epsilon_0}\frac{N^2-1}{N}\qquad 1=g_0^2\pi^4\frac{2}{\epsilon_0+\epsilon_1}. \tag{45}$$

In deriving this set of equations, the known relations for the SU(N) generators were used:

$$\lambda^a_{ij}\lambda^b_{jl}\lambda^a_{lk}=-\frac{2}{N}\lambda^a_{ik}\qquad \mathrm{Sp}\,\lambda^a\lambda^b=2\delta^{ab}$$

$$\lambda^a_{ik}\lambda^a_{kj}=\frac{2}{N}(N^2-1)\delta_{ij}.$$

It is easy to observe that the set obtained is one of linear homogeneous equations. A solution is possible only for N subject to the equation

$$N^2-2N-3=0.$$

The physically interesting solution is that with $N=3$. With $N=3$ we get the following values for the expansion parameters:

$$\epsilon_1=-\tfrac{2}{3}g_1^2 \qquad \epsilon_0=\tfrac{8}{3}g_1^2 \qquad g_1^2=\pi^4 g_0^2.$$

The field dimensions are

$$d=\tfrac{3}{2}+\tfrac{8}{3}g_1^2 \qquad \Delta=3+4g_1^2.$$

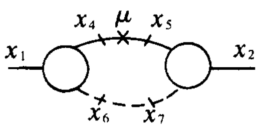
The validity of the calculations requires that

$$0<g_1^2\ll 1.$$

An interesting feature of the solution found in the three-vertex approximation under consideration is the fact that one of the parameters g_1^2 remains free. It must be determined after higher orders in ϵ_0 and ϵ_1 are taken into account. The next-order calculations are, however, made more complicated by the necessity to sum simultaneously to leading order in ϵ_1, an infinite number of diagrams which correspond to vertices with four boson fields. This, however, falls beyond the scope of the present work.

Appendix

Calculations of the three-vertex graphs in equations (34*b*) and (34*c*) are performed in the same way as in the case of scalar fields (Zaikin 1978). We list here only the final forms for the expressions:



$$= \int G^{dd\Delta}(x_1x_4x_6)(x_{45})_\mu G^{-1}(x_{45})G^{dd\Delta}(x_5x_2x_7)$$

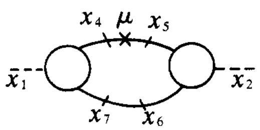
$$\times D_\Delta^{-1}(x_{67})\,\mathrm{d}^Dx_6\,\mathrm{d}^Dx_6\,\mathrm{d}^Dx_7\,\mathrm{d}^Dx_4$$

$$= g^2(\tfrac{1}{2}\Omega_D)^3\Gamma^2(h)\frac{\Gamma(d-h+\frac{1}{2})\Gamma(h-d+\frac{1}{2})}{\Gamma(D-d+\frac{1}{2})\Gamma(d+\frac{1}{2})}$$

$$\times\frac{\Gamma^2[\frac{1}{2}(D-\Delta)+\frac{1}{2}]\Gamma[\frac{1}{2}(D+\Delta-2d)]\Gamma[\frac{1}{2}(2D-2d-\Delta)]}{\Gamma^2(\frac{1}{2}\Delta+\frac{1}{2})\Gamma(d-\frac{1}{2}\Delta)\Gamma[\frac{1}{2}(d+\Delta-D)]}$$

$$\times\{\psi(d-\tfrac{1}{2}\Delta)+\psi[\tfrac{1}{2}(D+\Delta-2d)]+\psi(D-d-\tfrac{1}{2}\Delta)+\psi[\tfrac{1}{2}(2d+\Delta-D)]$$

$$-2\psi[\tfrac{1}{2}(D-\Delta)+\tfrac{1}{2}]-2\psi(\tfrac{1}{2}\Delta+\tfrac{1}{2})\}(x_{12})_\mu G_d(x_{12})$$



$$=\mathrm{Sp}\int G^{dd\Delta}(x_7x_4x_1)(x_{45})_\mu G_d^{-1}(x_{45})$$

$$\times G^{dd\Delta}(x_5x_6x_2)G^{-1}(x_{67})\,\mathrm{d}^Dx_4\,\mathrm{d}^Dx_5\,\mathrm{d}^Dx_6\,\mathrm{d}^Dx_7$$

$$= g^2(\tfrac{1}{2}\Omega_D)^3\Gamma^2(h)(x_{12})_\mu D_\Delta(x_{12})$$
$$\times\frac{\Gamma^2[\frac{1}{2}(D-\Delta)+\frac{1}{2}]\Gamma[\frac{1}{2}(D+\Delta-2d)]\Gamma(D-d-\frac{1}{2}\Delta)\Gamma(\Delta-h)\Gamma(h-\Delta)}{\Gamma^2(\frac{1}{2}\Delta+\frac{1}{2})\Gamma(d-\frac{1}{2}\Delta)\Gamma[\frac{1}{2}(2d+\Delta-D)]\Gamma(D-\Delta)\Gamma(\Delta)}$$
$$\times\{\psi[\tfrac{1}{2}(2d+\Delta-D)]+\psi(D-d-\tfrac{1}{2}\Delta)-\psi(d-\tfrac{1}{2}\Delta)-\psi[\tfrac{1}{2}(D+\Delta-2d)]\}.$$

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